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thanks

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Introduction

The theory of Stirling numbers was developed in 1749 by James Stirling (methodus differentialis) Stirling regarded these numbers not as representing the number of partition but rather as the coefficients of certain polynomials. The objective in this work is to study combinatorial principles especially, the principle of inclusion and exclusion and its application to count the number of Stirling of first, second kind and the total number of partitions of a set (Bell numbers). This work is diveded into three chapters. First we begin with Preliminaries from combinatorics (Combinatorial principles and generating function). In the combinatorics principles we use the multiplications principle, the addition principle and the principe of inclusion and exclusion. In the second chapter is concerned with partitions of numbers (Stirling numbers): Stirling numbers of first kind and second kind. The third chapter is devoted to study the total number of partitions of finite set (Bell number), partitions of an integer, recurrence relation for Bell numbers and finaly the Bell Triangle.

Chapter 1

Preliminaries from combinatorics

1.1 Combinatorial principles

1.1.1 The Multiplication principle

If $S_1, ..., S_n$ are non empty sets then the number of elements in the cartesian product

$$S_1 \times S_2, ..., \times S_n$$

is the product $\prod_{i=1}^{n} \mid S_i \mid$. That is

$$\mid S_1 \times S_2, ..., \times S_n \mid = \prod_{i=1}^{n} \mid S_i \mid .$$

we can reformulate the product rule in terms of events(jobs). If events $E_1, E_2, ..., E_n$ can happen

$$e_1, e_2, ...\text{and } e_n \text{ ways,}$$

respectively then the sequence of events E_1 first followed by E_1, ..., E_n can happen In

$$e_1.e_2....e_n$$

ways.

Example 1.1.1 *If* 2 *distinguishable dice are rolled, In how many ways can they fall ? If* 5 *distinguishable dice are rolled, how many possible "outcomes" are there ? How many if* 100 *distinguishable dice are fossed ?*

The first die can fall (event E_1) in 6 ways and the second can fall (event E_2) in 6 ways. Thus, there are

$$6.6 = 6^2 = 36.$$

Outcomes when 2 dice are rolled. If 5 distinguishable dice are rolled $\implies$ eath have 6 possibles outcomes so there are

$$6 \times 6 \times 6 \times 6 \times 6 = 6^5$$

Like wise there are 6^{100} possible outcomes when 100 dice are tossed.

Proposition 1.1.1 *Let $A_1, ..., A_n$ be a finite sets, the cardinality of $A_1 \times ... \times A_n$ is given by $|A_1|...|A_n|$.*

proof. *We proceed by induction on n. If $n = 1$, then the claim is obvious. Suppose for some n,*

$$|A_1 \times ... \times A_n| = |A_1|...|A_n|.$$

Consider the case with $n+1$ sets. Note that:

$$A_1 \times ... \times A_n \times A_{n+1} = (A_1 \times ... \times A_n) \times A_{n+1}.$$

$$|A_1 \times ... \times A_n| = |A_1|...|A_n|. \tag{1.1.1}$$

By inductive hypothesis (1.1.1)*, thus there are $|A_1|...|A_n|$ choices for the first n entries in the Cartesian product. There are $|A_{n+1}|$ choices for the final entry in the Cartesian product. Since this last entry is chosen independently from the others, it follows that*

$$|A_1 \times ... \times A_n \times A_{n+1}| = |A_1|...|A_n||A_{n+1}|.$$

Thus the proposition holds by the Principle of Mathematical Induction. ■

theorem 1.1.1 *(The Multiplication Principle) Suppose that there are n sets denoted $A_1, ..., A_n$. If elements can be selected from each set independently, then the number of ways to select one element from each set is given by $|A_1|...|A_n|$.*

proof. Note that this problem is equivalent to selecting an element from the set $A_1 \times ... \times A_n$. The cardinality of this set is $|A_1|...|A_n|$ by Proposition (1.1.1). ■

1.1.2 The Addition Principle

The principle of disjunctive counting (the whole is the sum of its parts) . If a set X is the union of disjoint non empty subsets $S_1, ..., S_n$; then

$$| X |=| S_1 | + | S_2 | +...+ | S_n |$$

proof. (the sets $S_1, S_2, ..., S_n$ must have no elements in common). Since

$$X = S_1 \cup S_2 \cup ... \cup S_n,$$

each element of X is in exactly one of the subsets S_i. In other words $S_1, S_2, ..., S_n$ is partition of X. Frequently, instead for the numbers of elements in a set , some problems ask for how many ways a certain event can happen. The diffrence is largely in semantics, for if A is an event, we can let X be the set of ways that A can be happen and count the number of elements in X. If $E_1, ..., E_n$ are mutually exclusive events, and

$$\begin{array}{l} E_1 \text{ can happen in } e_1 \text{ ways} \\ E_2 \text{ can happen in } e_2 \text{ ways} \\ \vdots \qquad\qquad\qquad \vdots \quad \vdots \\ E_n \text{ can happen in } e_n \text{ ways} \end{array}$$

then

$$E_1 \text{ or } E_2 \text{ or...or } E_n$$

can happen $e_1 + e_2 + ... + e_n$ ways. We have already looked at two rules dealing with the cardinality of the union of two sets. In particular, if A and B are disjoint sets, then

$$|A \cup B| = |A| + |B|.$$

As a generalization of this, if A and B are any sets then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

use the notation

$$\cup_{i=0}^{n} A_i = A_1 \cup ... \cup A_n$$

to denote the union of the sets $A_1, ..., A_n$. ■

theorem 1.1.2 *(The Generalized Addition Principle) If $A_1, ..., A_n$ are mutually disjoint sets, then:*

$$|\cup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i|.$$

proof. *We proceed by induction on n. If $n = 1$, then the claim is obvious. Assume that for some n, the mutually disjoint sets $A_1, ..., A_n$ satisfy:*

$$|\cup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i|.$$

Let $A_1, ..., A_n$, and A_{n+1} be mutually disjoint sets. Thus, $\cup_{i=1}^{n} A_i$ and A_{n+1} are disjoint sets. Hence, by definition, we have that

$$|\cup_{i=1}^{n+1} A_i| = |\cup_{i=0}^{n} A_i| + |A_{n+1}|.$$

Applying the inductive hypothes is yields

$$|\cup_{i=1}^{n+1} A_i| = |\cup_{i=0}^{n} A_i| + |A_{n+1}|.$$

Alternatively, if $x \in A_i$, then $x \notin A_j$ for $j \neq i$. Therefore, x is counted once by and once by $|\cup_{i=1}^{n} A_i|$ and once by

$$\sum_{i=1}^{n} |A_i|.$$

Similarly, if $x \notin A_i$ for all i, then x is counted zero times by both sides of the equation. Theorem can be summarized as follows: Suppose that there are n events. The i-th event can occur in a_i ways. If no two events can occur simultaneously (in other words, they are disjoint), then there are $a_i + ... + a_n$ ways that exactly one of the events can occur. ■

Example 1.1.2 *Suppose that you can go to one of five small restaurants for dinner. No two restaurants offer the same menu items. The first restaurant offers 8 menu items, the second restaurant offers 6 items, the third restaurant offers 11 items, the fourth restaurant offers 4 items, and the final restaurant offers 20 items. How many meals are possible ?*

Let A_i be the set of all menu choices in the i-th restaurant. Since no two restaurants offer the same menu items, these sets are disjoint. So by the Addition Principle, we have

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| &= |A_1| + |A_2| + |A_3| + |A_4| + |A_5| \\ &= 8 + 6 + 11 + 4 + 20 = 49.\end{aligned}$$

1.1.3 Principle of Inclusion and Exclusion

In the sum

$$|A_1| + |A_2| + |A_3|,$$

an element of $A_1 \cap A_2$ is included in at least two terms; an element of

$$A_1 \cap A_2 \cap A_3$$

is included in all three terms. In the sum

$$|A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|, \tag{1.1.2}$$

an element in A_1, but not A_2 or A_3, is included in only one term. An element in A_1 and A_2 , but not in A_3 , is included in two positive terms and one negative term, so it is counted exactly once in this formula. An element in A_1, A_2, and A_3 is included in three positive terms and three negative terms, so it is not counted at all in expression 1.1.2. Therefore, the expression in 1.1.2 counts once each element of $A_1 U A_2 U A_3$ except for the elements in $A_1 \cap A_2 \cap A_3$

$$\begin{aligned}|A_1 U A_2 U A_3| &= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| \\ &\quad - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|. \qquad (1.1.3)\end{aligned}$$

A formula such as 1.1.3 is called an inclusion-exclusion formula, and the argument that led to it is called an inclusion-exclusion argument.

Example 1.1.3 *Find the number S of integers included between 1 and 200, and divisible by $2, 3$ or 5 (non-exclusive way). For $i \in \{1, 3, 5\}$, let's note the set of integers at most equal to 200 and divisible by i.*

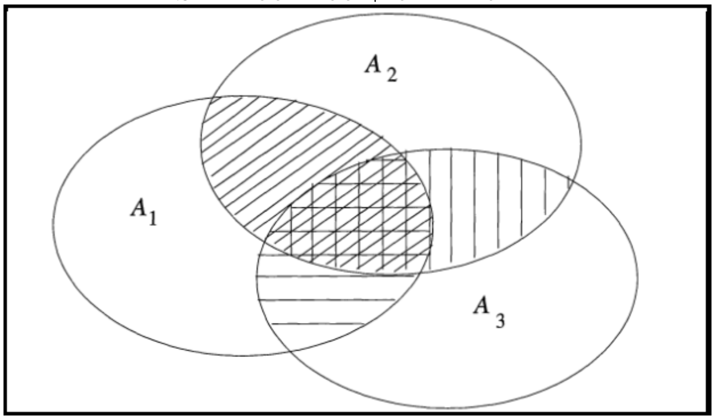
$$\begin{aligned} S &= |A_2 \cup A_3 \cup A_5| \\ &= |A_2| + |A_3| + |A_5| - |A_2 \cap A_3| - |A_2 \cap A_5| - |A_3 \cap A_5| + |A_2 \cap A_3 \cap A_5| \end{aligned}$$

Or $|A_2| = [\frac{200}{2}] = 100$, $|A_3| = [\frac{200}{3}] = 66$, $|A_5| = [\frac{200}{5}] = 40$.

more are 2 and 3 prime betwen them 2 and 3 is, $A_2 \cap A_3$ is the set of multiples of 6 included betwen 1 and 200, and so $|A_2 \cap A_3| = [\frac{200}{6}] = 33$.

of the same $|A_2 \cap A_5| = [\frac{200}{10}]$, $|A_3 \cap A_5| = [\frac{200}{15}] = 13$ et $|A_2 \cap A_3 \cap A_5| = [\frac{200}{30}] = 6$.

From where $S = 206 - 66 + 6 = 146$.



corollary 1.1.1 *For all $n \in \mathbb{N}$,*

$\binom{n}{0} + \binom{n}{2} + ... = \binom{n}{1} + \binom{n}{3} + ...$

proof. *Let $x = -1$ and $y = 1$. Note that $x + y = 0$. Thus by the Binomial Theorem, we have:*

$$\begin{aligned} 0 &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \\ &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \pm ... \\ &\Rightarrow \binom{n}{0} + \binom{n}{2} + ... = \binom{n}{1} + \binom{n}{3} + .. \end{aligned}$$

■

theorem 1.1.3 *(The Generalized Principle of Inclusion and Exclusion) We define the quantity c_m for $m = 1, ..., n$ as*

$$c_m \;=\; \sum_{i_1,\ldots,i_m} \mid A_{i_1} \cap \ldots \cap A_{i_m} \mid$$

Let $A_1, \ldots, A_n$ be subsets of some universal (or relevant) set A. Then the following hold:

$$\begin{aligned} \mid \quad & A_1 \cup \ldots \cup A_n \mid = c_1 - c_2 + \ldots + (-1)^{l+1} c_l + \ldots + (-1)^{n+1} c_n \\ = \quad & \sum_{l=1}^{n} (-1)^{l+1} c_l \end{aligned}$$

proof. Let $x \in A_1 \cup \ldots \cup A_n$. It suffices to show that x is counted the same number of times by both sides of the equation. Clearly, x is counted once in $|A_1 \cup \ldots \cup A_n|$. Suppose that x is an element of exactly m of the subsets. It follows that:

(a) x is counted m times by c_1.

(b) x is counted $\binom{m}{2}$ times by c_2.

(c) In general, x is counted $\binom{m}{l}$ times by c_l .

From this it follows that the right side of the equation counts x

$$\binom{m}{1} - \binom{m}{2} + \ldots + (-1)^{l+1} \binom{m}{l} + \ldots + (-1)^{m+1} \binom{m}{m}$$

$$= \sum_{i=1}^{m} (-1)^{i+1} \binom{m}{i}$$

times. Recall that Corollary implies that

$$0 = \sum_{i=0}^{m} (-1)^i \binom{m}{i} = 1 + \sum_{i=1}^{m} (-1)^i \binom{m}{i}$$

$$\Longrightarrow\; 1 = -\sum_{i=1}^{m} (-1)^i \binom{m}{i} = \sum_{i=1}^{m} (-1)^{i+1} \binom{m}{i}.$$

Since x is counted the same number of times in each side of the equation, the theorem holds.

■

1.2 Relations and Partitions

definition 1.2.1 *Let $S \neq \varnothing$ be a set. A partition of S is a collection of non-empty, pairwise disjoint subsets of S whose union is S.*

Example 1.2.1 *Let*

$$2\mathbb{Z} = \{..., -6, -4, -2, 0, 2, 4, 6, ...\} = \overline{0}$$

be the set of even integers and let

$$2\mathbb{Z} + 1 = \{..., -5, -3, -1, 1, 3, 5, ...\} = \overline{1}$$

be the set of odd integers. Then

$$(2\mathbb{Z}) \cup (2\mathbb{Z} + 1) = \mathbb{Z}, (2\mathbb{Z}) \cap (2\mathbb{Z} + 1) = \emptyset,$$

and so $\{2\mathbb{Z}, 2\mathbb{Z} + 1\}$ is a partition of $\mathbb{Z}$.

Example 1.2.2 *Let*

$$3\mathbb{Z} = \{... - 9, , -6, -3, 0, 3, 6, 9, ...\} = \overline{0}$$

be the integral multiples of 3, let

$$3\mathbb{Z} + 1 = \{..., -8, -5, -2, 1, 4, 7, ...\} = \overline{1}$$

be the integers leaving remainder 1 upon division by 3, and let

$$3\mathbb{Z} + 2 = \{..., -7, -4, -1, 2, 5, 8, ...\} = \overline{2}$$

be integers leaving remainder 2 upon division by 3. Then

$$(3\mathbb{Z}) \cup (3\mathbb{Z} + 1) \cup (3\mathbb{Z} + 2) = \mathbb{Z},$$

$$(3\mathbb{Z}) \cap (3\mathbb{Z} + 1) = \emptyset, (3\mathbb{Z}) \cap (3\mathbb{Z} + 2) = \emptyset, (3\mathbb{Z} + 1) \cap (3\mathbb{Z} + 2) = \emptyset,$$

and so $\{3\mathbb{Z},\ 3\mathbb{Z} + 1,\ 3\mathbb{Z} + 2\}$ is a partition of $\mathbb{Z}$.

definition 1.2.2 *Let $\sim$ be a relation on a non-empty set A. Then $\sim$ is said to form an equivalence relation if $\sim$ is reflexive, symmetric and transitive. The equivalence class containing $a \in A$, denoted $[a]$, is defined as*

$$[a] := \{b \in A : b \sim a\}$$

Example 1.2.3 *Let* S *={All Human Beings}, and define* $\sim$ *on* S *as* $a \sim b$ *if and only if* a *and* b *have the same mother. Then* $a \sim a$ *since any human* a *has the same mother as himself. Similarly,* $a \sim b \Rightarrow b \sim a$ *and* $(a \sim b)$ *and* $(b \sim c) \Rightarrow (a \sim c)$*. Therefore* $\sim$ *is an equivalence relation.*

definition 1.2.3 ***Example 1.2.4*** *The partitions of* $A = \{a, b, c, d\}$ *into* 3*-parts are* $a|b|cd, a|bc|d, ac|b|d, a|$ *where the expression* $a|bc|d$ *represents the partition* $A_1 = \{a\}$*,* $A_2 = \{b, c\}$ *and* $A_3 = \{d\}$*.*

1.3 Generating Functions

definition 1.3.1 *Generating functions are a general mathematical tool developed by de Moivre, Stirling, and Euler in the* 18 *th century, are used often in combinatorics, and are algebraic objects that provide a powerful tool for analyzing recurrence relations. As usual, we start by taking a concrete example: In how many ways can you make change for a dollar? We'll assume that we're dealing with only five types of coins—pennies, nickels, dimes, quarters, and half dollars.*

definition 1.3.2 *What is a generating function ?*

Consider the polynomial

$$(x + 1)^n = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0.$$

We know that $a_i = \binom{n}{i}$, the number of i subsets of an n-set.

Consider the polynomial x falling k, or

$$(x)_k = x(x - 1)...(x - k + 1) = a_k x^k + a_{k-1} x^{k-1} + ... + a_0.$$

The a_i are the Stirling numbers of the first kind $s(k; k)$, $s(k; k - 1), ..., s(k; 0)$, with $a_i = s(k; i)$. Polynomials have

finitely many non-zero coeffcients. Consider instead the expression

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + ...$$

We could say that it is

$$\sum_{i=0}^{\infty} a_i x^i$$

with $a_i = 1 + i$ for every $i = 0, 1, 2, ...$.

The expression

$$1 + 2x + 3x^2 + ...$$

is called the generating function for the series $a_i = i + 1$ for all $i \geq 0$.

definition 1.3.3 *If $a_m, a_{m+1}, ...$is an infinite sequence with m some integer (positive, zero, or negative) then the expression*

$$\sum_{i=m}^{\infty} a_i x^i$$

is the generating function for the sequence $a_m, a_{m+1}, ...$

1.3.1 Rational generating functions

definition 1.3.4 *Given any sequence $a_0, a_1, a_2, ...,$ the ordinary generating function is*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

The generating function $f(x)$ contains all of the information about the sequence $\{a_n\}$, and, being an algebraic entity, it is often easier to manipulate than the sequence itself. The term a_n is recovered by finding the coefficient of x^n in $f(x)$.

1.3.2 Operations on generating functions

definition 1.3.5 *We can sum and multiply generating functions. Given two functions $f(x)$ and $g(x)$ associated to the sequences $(a_0, a_1, a_2, ...)$ and $(b_0, b_1, b_2, ...)$, respectively , the sum and product are defined by*

$$(f + g)(x) = f(x) + g(x) = (a_{0+}b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + ...,$$

$$(fg)(x) = f(x)g(x) = (a_0b_0) + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + ...$$

In the product, the c_k cofficient of x^k is

$$c_k = a_kb_0 + a_{k-1}b_1 + a_{k-2}b_2 + ... + a_1b_{k-1} + a_0b_k.$$

With these rules, generating functions behave like infinite polynomials.

1.3.3 Counting with generating functions

Example 1.3.1 *To find the ordinary generating function for the sequence*

$$\prec 1, 1, 1, ..., 1, ... \succ$$

we have

$$\begin{aligned} G(x) &= \sum_{i=0}^{\infty} X^i = X^0 + X^1 + X^2 + ... + X^n + ... \\ &= \frac{1}{1-X}. \end{aligned}$$

namely for $-1 < X < 1$ (and can be proven using the formula for the geometric series).

Example 1.3.2

$$\prec 1, -1, 1, -1, 1, -1, ..., (-1)^i \succ$$

$$\begin{aligned} G(x) &= \sum_{i=0}^{\infty} (-1)^i X^i = X^0 - X^1 + X^2 - X^3 + X^4 - X^5. \\ &= \frac{1}{1+X}. \end{aligned}$$

Example 1.3.3 *girl can choose two items fruits a basket containing an apple, an orange, a pear, a banana, and a papaya. How many way can this be done ?*

$$C(5, 2) = \frac{5!}{(5-2)!2!} = \frac{5!}{3!2!} = 10$$

ways to choose two of them. using generating function. We can chooses

$$(0\ apples{+}1\ apples)(0\ oranges{+}1\ oranges)(0\ pear{+}1\ pear)(0\ banana{+}1\ banana)(0\ papaya{+}1\ papaya)$$

$$\begin{aligned}(X^0+X^1)(X^0+X^1)(X^0+X^1)(X^0+X^1)(X^0+X^1) &= (1+X)^5 \\ &= 1+5X+10X^2+10X^3+5X^4+X^5\end{aligned}$$

There are 10 ways to choose two of them .

Chapter 2

Partitions numbers

2.1 Partitions numbers

definition 2.1.1 *we defined the partition number $p(n, k)$ to be the number of ways to write n as a sum of k positive integers. We can also think of $p(n, k)$ as the number of onto functions*

$$f : X- > Y, \quad | X |= n, \quad | Y |= k,$$

where X and Y are unlabeled sets. Also, the partition number $p(n)$ has been defined as $p(n) = \sum_{k=1}^{n} p(n, k)$ this section we develop generating functions for partition numbers.

Example 2.1.1 *Determine $p(4, k)$, for $k = l, 2, 3, 4$, and $p(4)$.*

We have

$$\begin{aligned} P(4,1) &= 1 \ (4 = 4) \\ P(4,2) &= 2 \ (2 + 2 = 4, 3 + 1 = 4) \\ P(4,3) &= 1 \ (2 + 1 + 1 = 4) \\ p(4,4) &= 1 \ (1 + 1 + 1 + 1 = 4) \end{aligned}$$

and

$$
\begin{aligned}
&= p(4,1)+p(4,2)+p(4,3)+p(4,4)\\
&= 1+2+1+1\\
&= 5
\end{aligned}
$$

Table the number of partitions of n into k parts

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	1	1							
4	1	2	1	1						
5	1	2	2	1	1					
6	1	3	3	2	1	1				
7	1	3	4	3	2	1	1			
8	1	4	5	5	3	2	1	1		
9	1	4	7	6	5	3	2	1	1	
10	1	5	8	9	7	5	3	2	1	1

2.2 Stirling numbers of the first kind

Let n and k be positive integers whith $k \leq n$.

definition 2.2.1 *The stirling number of the first kind, $s(n,k)$, is defined by the rule that $(-1)^{n-k}s(n,k)$ is the number of permutations of $\{1,...,n\}$ with k cycles. (Note the sign sometimes a diffrent convention is used, according to which the stirling numbers are the absolute values of those defined here.)*

2.3 Stirling Numbers of the First Kind as Polynomial Coefficients

Stirling regarded these numbers not as representing the number of partitions but rather as the coefficients of a certain polynomial. To see why, we first examine the Stirling numbers of the first kind. These arise from studying polynomials and at first glance appear to have nothing to do with partitions. Suppose m and n are non negative integers and let $m \leq n$. Recall that the number $\frac{n!}{(n-m)!}$ of one-to-one functions from an m-element set to a n-element set is denoted by $(n)_m$.

$$(n)_m = n(n-1)...(n-m+1),$$

which is a polynomial of degree m in n. Because it is a polynomial, there are numbers $s(n,j)$ such that it can be written in the form m

$$(n)_m = \sum_{j=0}^{m} s(m,j)n^j.$$

We use this equation to define Stirling numbers of the first kind. In other words, the numbers $s(m,j)$ are the coefficients of n^j that arise when we expand the product $(n)_m$. The numbers $s(m,j)$ are called **Stirling numbers of the first kind**. For any real or even complex number x, we can define

$$(x)_m = x(x-1)...(x-m+1).$$

(We define $(x)_0 = 1$.) Then $(x)_m$ is a polynomial in x of degree m. We call $(x)_m$ a **factorial power** of x of degree m. Although, technically speaking, we defined $s(m,j)$ in terms of the coefficients of a polynomial with an integral variable, it is not surprising that changing n to x does not change the coefficients $s(m,j)$.

theorem 2.3.1 *For any x,*

$$(x)_m = \sum_{j=0}^{m} s(m,j)x^j.$$

Example 2.3.1 *Write $(x)_3$ as an ordinary polynomial.*

$$(x)_3 = x(x-1)(x-2) = x(x^2-3x+2)$$

$$= x^3 - 3x^2 + 2x.$$

Note that we can tell from this example that $s(3,0) = 0$, $s(3,1) = 2$, $s(3,2) = -3$, *and* $s(3,3) = 1$.

Proposition 2.3.1 $s(n+1,k) = -ns(n,k) + s(n,k-1)$

Proposition 2.3.2 $\sum_{k=1}^{n} (-1)^{n-k} s(n,k) = \sum_{k=1}^{n} \mid s(n,k) \mid = n!$

2.4 Stirling numbers of the second kind

definition 2.4.1 *The stirling number of the second kind,* $S(n,k)$, *is the number of partitions of* $\{1, ..., n\}$ *with* k *(non-empty) parts.*

The definitions can be extended to all n and k by defining the stirling numbers to be 0 unless $1 \leq k \leq n$. Both arrays satisfy recurrence relations, similar to that for pascal's triangle. Recall that $s(n,0) = S(n,0) = 0$ for all n. (In combinatorics, the **Stirling numbers of the second kind** tell us how many ways there are of dividing up a set of n objects (all different ,or at least all labeled) into K non empty subsets . The K subsets aren't labeled.)

We write Stirling numbers of the second kind as $S(n,k)$ or :

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \text{ or } S_k^n.$$

2.5 Stirling's Triangle of the Second Kind

theorem 2.5.1 $S(m,n) = S(m-1,n-1) + nS(m-1,n)$.

proof. Let $M = \{1, 2, ..., m\}$ and $M\prime = \{1, 2, ..., m-1\}$. Then a partition of M may have one class consisting of m alone; the number of such partitions is the number of partitions of $M\prime$ with $n-1$ parts. A partition of M may have m in a class containing some elements of $M\prime$ as well. Deleting m gives a partition of $M\prime$ with n parts, and since m may have been in any of the n parts, n different partitions of M yield the same partition of $M\prime$. Since m either is or isn't in a class by itself, this proves the theorem. ■

Example 2.5.1 *It is customary to write Stirling's triangle in the form of a right triangle as in the portion on and below the main diagonal in Table 3.1. Note that*

$$S(3,2) = S(2,1) + 2S(2,2) = 1 + 2.1 = 3$$

says that $S(3,2)$ is the sum of twice the number above it plus the number immediately to the left of the number directly above it. The remainder of the table is filled in similarly.

The second recursion formula for S(m, n)

Proposition 2.5.1 *For $m > 0$, $S(m,n) = \sum_{j=0}^{m-1} \binom{m-1}{j} S(j, n-1)$*

proof. If P is a partition of $M = \{1, 2, ..., m\}$ and if we eliminate from P the class containing m, we get a partition of a subset $J \sqsubseteq \{1, 2, ..., m-1\}$. The resulting partition has $n-1$ parts. Every partition of every subset J of $\{1, 2, \ldots , m-1\}$ into $n-1$ parts arises exactly once from this kind of construction, so the number of partitions of M into n parts is the number of partitions of subsets of $\{1, 2, ..., m-1\}$ into $n-1$ parts. The formula totals the number of such partitions of a subset J for all possible sizes of J. ■

As you can see by examining Table 3.1,

$$S(4,3) = \sum_{j=0}^{3} \binom{3}{0} S(j,2) = 1.0 + 3.0 + 3.1 + 1.3 = 6$$

2.6 The Inverse Image Partition

The numbers $S(m,n)$ are closely related to functions. Suppose that we have a function f from $M = \{1, 2, ..., m\}$ onto $N = \{1, 2, ..., n\}$. We use the notation

$$f^{-1}(i) = \{x \mid f(x) = i\};$$

that is, $f^{-1}(i)$ is the subset of M consisting of elements that f maps onto i. The set $f^{-1}(i)$ is called the **inverse image** of i. Thus, we have a partition, called the **inverse image partition**,

$$P = \{f^{-1}(1), f^{-1}(2), ..., f^{-1}(n)\}$$

of M into n classes. This is the equivalence class partition of the inverse image relation $R(f)$ associated with f. In addition, we have a function g defined on P by $g(f^{-1}(i)) = i$; note that g is a one-to-one function from P to N.

2.6.1 Onto Functions and Stirling Numbers

On the other hand, given a partition

$$Q = \{C_1, C_2, ..., C_n\}$$

of M into n classes and a one-to-one function g from Q to N, we can define a function f from M onto N by

$$f(x) = i \text{ if } x \text{ is in } C_i.$$

Thus, the number $F(M,N)$ of functions from M onto N is, by the product principle, $S(m,n)n!$, the product of the number of partitions and the number of one-to-one functions. This proves the following theorem.

theorem 2.6.1 *The number of functions from an m-element set onto a n-element set is $S(m,n)n!$.*

Proposition 2.6.1 $S(n,1) = 1$ *for* $n \geq 1$.

$S(n,n) = 1$ for $n \geq 1$.
$S(n,k) = (n-1,k-1) + k(n-1,k)$ for $2 \leq k < n$.

From the recurrence formula, we obtain a table (Table 3.1) of values of $S(n,k)$for small n and k.

$S(n,k)$	$k=1$	2	3	4	5	6
$n=1$	1	0	0	0	0	0
$n=2$	1	1	0	0	0	0
$n=3$	1	3	1	0	0	0
$n=4$	1	7	6	1	0	0
$n=5$	1	15	25	10	1	0
$n=6$	1	31	90	65	15	1

table(3,1)

Example 2.6.1 *Let us verify an entry of the table, say $S(4,3) = 6$. There are six ways to partition of the set $\{1,2,3,4\}$ into three subsets:* $\{12,3,4\}$, $\{1,3,24\}$, $\{1,2,34\}$, $\{13,2,4\}$, $\{1,4,23\}$, $\{14,2,3\}$

Example 2.6.2 *The set* $\{a, b, c\}$ *can be divided into* $\{a, b\}$ *and* $\{c\}$*, or* $\{a, c\}$ *and* $\{b\}$*, or* $\{b, c\}$*and* $\{c\}$*. So* $S(2, 3) = 3$.

Example 2.6.3 *How many partitions of the set* $\{1, 2, 3, 4\}$ *are there ?*

In a partition of a set of four elements, the sizes of the equivalence classes sum to 4.There are five possibilities for these sizes:

$$4$$

$$3 + 1$$

$$2 + 2$$

$$2 + 1 + 1$$

$$1 + 1 + 1 + 1.$$

Example 2.6.4 *For example, the partition*

$$\{\{1, 2\}, \{4\}, \{5\}\}$$

is of type $2 + 1 + 1$*. It is an easy matter to count the partitions of each type, obtaining, respectively,* $1, 4, 3, 6,$ *and* 1*, for a total of* 15*.*

theorem 2.6.2 $s(n, n) = S(n, n) = 1$.

Proposition 2.6.2 $S(n + 1, k) = S(n, k - 1) + kS(n, k)$

proof. *For example, suppose you step into a room with* n *other people. All* $n + 1$ *of you are to be separated into* k *non-empty groups. By definition there are* $S(n + 1, k)$ *ways this can be accomplished. By approaching the matter from a different direction, however, we can get another formula, which must therefore be equal to* $S(n + 1, k)$*. There are two possibilities. First, you could be antisocial and form a group all by yourself. The other* n *people would then have to form* $k - 1$ *groups. There are* $S(n, k - 1)$ *ways for them to do this. Alternatively, you could decide you feel like having company. In this case the other* n *people would form* k *groups, and you would then join one of their groups. There are* k *choices for* *which group you decide to join, and $S(n,k)$ways for the other people to have formed the k groups, for a total of $kS(n,k)$ possibilities. Adding the antisocial case, we find*

$$S(n+1,k) = S(n,k-1) + kS(n,k).$$

■

2.6.2 The number of surjective functions

corollary 2.6.1 *The number of surjective mappings from an n-set to a k-set is given by*

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n \, .$$

In particular, we have

$$n! = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^n \, .$$

We take X to be the set of all mappings from $\{1,...,n\}$ to $\{1,...,k\}$, so that $\mid X \mid = k^n$. For $i = 1,...,k$, we let A_i be the set of mappings f for which the point i does not lie in the range of f. Then each $f(x)$ can be any of the $k-1$ points different from i, and so $|A_i| = (k-1)^n$. More generally, A_I consists of all mappings whose range contains no point of I, and $|A_I| = (k-|I|)^n$.

A mapping is a surjection if and only if it lies in none of the sets A_i, So, by PIE (Principle of Inclusion and Exclusion), the number of surjections is equal to

$$\sum_{I \subseteq \{1,...,k\}} (-1)^{|I|} (k-|I|)^n.$$

Put $i =\mid I \mid$. There are $\binom{k}{i}$ sets I of cardinality i, where i runs from 1 to k; this gives the result. If $k = n$, then the permutations of $\{1,...,n\}$ are precisely the surjective mappings from this set to itself.

Proposition 2.6.3 $S(n,k) = \frac{1}{k!} \sum\limits_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n$.

proof. *The numbers $S(m,n)$ (which we defined as the number of partitions of m-element set into n classes) are called Stirling numbers of the second. We saw in the last section that* *this expression, without the factor* $\frac{1}{k!}$*, is the number of surjections from* $\{1,...,n\}$ *to* $\{1,...,k\}$*. (I have also replaced the dummy variable* i *by* $j = k - i$*, and dropped the term with* $j = 0$*.) So it suffices to prove that the number of surjections is* $k!S(n,k)$*. Each surjection* f *defines a partition of* $\{1,...,n\}$ *with* k *non-empty parts, viz.,* $f^{-1}(1),...,f^{-1}(k)$*. But every partition arises from exactly* $k!$ *surjections, since we may assign the numbers* $1,...,k$ *to the parts in any order. The result is proved.* . ■

2.7 Stirling Numbers of the Second Kind as Polynomial Coefficients

definition 2.7.1 *The numbers* $S(m,n)$ *(which we defined as the number of partitions of an* m*-element set into* n *classes) are called Stirling numbers of the second kind because Stirling discovered they also relate the factorial powers of* n *to ordinary powers of* n*.*

theorem 2.7.1 $n^m = \sum_{j=0}^{n} S(m,j)(n)_j$.

proof. *We begin with three observations. we know that for a particular set* J*, there are* $S(m,j)j!$ *functions from the* n*- element set onto* J*. We also know that every function from* M *to* N *maps onto some subset* J *of* N*. Third, we know there are* $\binom{n}{j}$ *different subsets* J *of size* j *that a function can map* M *onto. Using the sum and product principle, we put these three ideas together to get*

$$\begin{aligned} n^m &= \sum_{j=0}^{n} \binom{n}{j} S(m,j)j! \\ &= \sum_{j=0}^{n} \frac{n!}{(n-j)!} S(m,j) \\ &= \sum_{j=0}^{n} S(m,j)(n)_j. \end{aligned}$$

■

Notation 2.7.1 *Our definition that* $S(0,0) = 1$ *makes this theorem true for* $m = 0$ *as well.*

theorem 2.7.2 $x^m = \sum_{j=0}^{n} S(m,j)(x)_j$.

Example 2.7.1 *Write x^3 as a combination of falling factorials*

Example 2.7.2 *Applying Theorem we get*

$$\begin{aligned} x^3 &= \sum_{j=0}^{n} S(3,j)(x)_j \\ &= S(3,0)(x)_0 + S(3,1)(x)_1 + S(3,2)(x)_2 + S(3,3) \\ &= 0 + 1(x) + 3(x)_2 + 1(x)_3 \\ &= x + 3(x)(x-1) + 1(x)(x-1)(x-2). \end{aligned}$$

Example 2.7.3 *To verify the use of the theorem, we multiply out the polynomials and add:*

$$x + 3(x^2 - x) + 1.x(x^2 - 3x + 2) = x + 3x^2 - 3x + x^3 - 3x^2 + 2x = x^3.$$

Example 2.7.4 *The portion on and below the main diagonal in Table 3.1. Note that*

$$S(3,2) = S(2,1) + 2S(2,2) = 1 + 2.1 = 3$$

says that $S(3,2)$ is the sum of twice the number above it plus the number immediately to the left of the number directly above it. The remainder of the table is filled in similarly.

Proposition 2.7.1 $t^n = \sum\limits_{k=1}^{n} S(n,k)(t)_k$.

proof. $t^n = \sum\limits_{k=1}^{n} S(n,k)(t)_k$ then

$$t^{n+1} = t^n.t = \sum_{k=1}^{n} (t)_k((t-k)+k)S(n,k)$$

Since $(t)_k(t-k) = (t)_{k+1}$, we have

$$\begin{aligned} t^{n+1} &= \sum_{k=1}^{n} S(n,k)(t)_{k+1} + \sum_{k=1}^{n} kS(n,k) \\ &= \sum_{k=1}^{n+1} (S(n,k-1) + kS(n,k))(t)_k \\ &= \sum_{k=1}^{n+1} S(n+1,k)(t) \end{aligned}$$

Since

$$S(n,0) = S(n,n+1) = 0$$

■

Chapter 3

The Total Number of Partitions of a Set

3.1 The Total Number of Partitions of a Set

The total number B_m of partitions of an m-element set is a sum of Stirling numbers of the second kind, namely

$$B_m = \sum_{n=0}^{m} S(m, n).$$

The numbers B_m are called the **Bell numbers** after E. T. Bell who studied how the value of B_m increases as m increases. Much of the information known about the Bell numbers can be derived from properties of the Stirling numbers.

For example, from the second recursion formula for $S(m, n)$, we get the following theorem.

theorem 3.1.1 *For* $m > 0$, $B_m = \sum_{j=0}^{m-1} \binom{m-1}{j} B_j$.

proof. since

$$B_m = \sum_{n=0}^{m} S(m, n),$$

$$
\begin{aligned}
B_m &= \sum_{n=0}^{m}\sum_{j=0}^{m-1}\binom{m-1}{j}S(j,n-1)\\
&= \sum_{j=0}^{m-1}\sum_{n=0}^{m}\binom{m-1}{j}S(j,n-1)\\
&= \sum_{j=0}^{m-1}\binom{m-1}{j}\sum_{n=0}^{m}S(j,n-1)\\
&= \sum_{j=0}^{m-1}\binom{m-1}{j}\sum_{n=0}^{m-1}S(j,i),
\end{aligned}
$$

since $S(j,-1) = 0$. However,

$$\sum_{i=0}^{m-1}S(j,i) = \sum_{i=0}^{j}S(j,i) = B_j$$

■

Example 3.1.1 *The n-th Bell number, denoted $B(n)$, is the total number of partitions of the set $\{1, 2, 3, ..., n\}$. The Stirling number of the second kind $\left\{ {n \atop k} \right\}$ is the number of partitions of $\{1, 2, 3, ..., n\}$ into k equivalence classes. The partition number $p(n)$ is the total number of partitions of a set of n indistinguishable elements. These are also called partitions of an integer. According to the above example, $B(4) = 15$, $\left\{ {4 \atop 1} \right\} = 1$, $\left\{ {4 \atop 2} \right\} = 7$, $\left\{ {4 \atop 3} \right\} = 6$, $\left\{ {4 \atop 4} \right\} = 1$, and $p(4) = 5$. We also define $p(n, k)$ to be the number of partitions of n indistinguishable objects into k parts. From the above example, $p(4, 1) = 1$, $p(4, 2) = 2$, $p(4, 3) = 1$, and $p(4, 4) = 1$.*

Proposition 3.1.1 $\sum\limits_{k=1}^{n} S(n,k) = B_n$, *where B_n is the n^{th} Bell number*

Example 3.1.2 *For example, $B_3 = 5$; the five partitions of {1,2,3} are*

$$\{\{1, 2, 3\}\}$$
$$\{\{1, 2\}, \{3\}\}$$
$$\{\{1, 3\}, \{2\}\}$$
$$\{\{2, 3\}, \{1\}\}$$
$$\{\{1\}, \{2\}, \{3\}\}$$

Similarly, $B_2 = 2$, $B_1 = 1$. What is B_0 ? Since the parts of a partition are non-empty by definition, a partition of the empty set can not have any parts at all, and must be the empty set. But the empty set is indeed a partition of itself! So $B_0 = 1$.

3.2 Partitions of an integer

A partition of a positive integer k is a way to write k as a sum of positive integers. For example, the partitions of 5 are the following:

$$\begin{array}{ll} 5 & \\ 4+1 & 3+2 \\ 3+1+1 & 2+2+1 \\ 2+1+1+1 & \\ 1+1+1+1+1 & \end{array}$$

The above listing of the partitions of 5 are grouped according to the size of the partition ($p(5) = 7$). $3 + 1 + 1$ and $2 + 2 + 1$ are the partitions of 5 into three parts. For every positive integer n there will be only one partition into one part, namely n itself, andonly one partition into n parts, namely $1 + 1 + ... + 1$.

Example 3.2.1 *In counting partitions we have indicated that order is not important, so there are three different partitions of* 6 *with* 3 *summands:* $4+1+1$, $3+2+1$, *and* $2+2+2$.

definition 3.2.1 *Bell numbers is the number of partitions of the set to* n *elementes (we take by convention* $B_0 = 1$*)*

$$\begin{array}{ll} B_1 = 1 & \\ B_2 = 2 & (1,2) \text{ et } ((1),(2)) \\ B_3 = 5 & (1,2,3),((1),(2),(3)),((1,2),(3)),((1,3),(2)),((2,3),(1)). \end{array}$$

3.2.1 Recurrence for Bell numbers

For $n \geq 1$,

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

Example 3.2.2

$$\begin{aligned} B_4 &= \binom{3}{3}B_3 + \binom{3}{2}B_2 + \binom{3}{1}B_1 + \binom{3}{0}B_0 \\ &= 1 + (3.1) + (3.2) + 5 \\ &= 15 \end{aligned}$$

Example 3.2.3 *We apply the recurrence to compute the first few Bell numbers:*

$B_1 = \sum_{k=0}^{0} \binom{0}{0} B_0 = 1.1 = 1$

$B_2 = \sum_{k=0}^{1} \binom{1}{k} B_1 = \binom{1}{0} B_0 + \binom{1}{1} B_1 = 1.1 + 1.1 = 1 + 1 = 2$

$B_3 = \sum_{k=0}^{2} \binom{2}{k} B_k = 1.1 + 2.1 + 1.2 = 5$

$B_4 = \sum_{k=0}^{3} \binom{3}{k} B_k = 1.1 + 3.1 + 3.2 + 1.5 = 15$

The Bell numbers grow exponentially fast; the first few are 1, 1 , 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, 27644437.

3.3 Bell's numbers and the Bell Triangle

Bell's numbers and the Bell Triangle (some times called the Pierce triangle or Aitken array) are a sequence of numbers which count the possible partitions of a set, and the Triangle which makes derivations of them easy.

The first Bell numbers are:

1, 1 , 2, 5, 15, 52, 203, 877, 4140, 21147, 115975. The n-th Bell number, B_n the number of non empty subsets a set of size n can be partitioned into $B_0 = 1$; there is just one possible partitioned of the set containing member, so $B_1 = 1$.

There are two partitions possible for a set with two elements, so $B_2 = 2$ is just two.

A set with three elements can be partioned five ways. Taking the set $\{a, b, c\}$, the five possible partitions are:

$\{(a)\,(b)\,(c)\}, \{(a,b)\,,(c)\}, \{(a,c)\,(b)\}, \{(b,c)\,(a)\}, \{(a,b,c)\}.$

There isn't any simple formula that can give us B_n but we can find Bell's numbers in the Bell Triangle, or use the following recursive equation to define them:

The Bell Triangle is an easy-to-fill in right triangle which gives us, in the left hand column, all of the Bell numbers.

3.3.1 Creating the Triangle

We make the triangle in this way:

On row one, write the number 1 Begin all other rows with the last number of the previous row. The last number in row 1 was 1, so row 2 also begins with 1. All other numbers are found by adding the last number to the one above it. To find out what to write in the second place in row 2, we look at the place before it, place 1. This is just 1, and the digit 1 is above it, so $1+1=2$.

The first part of the Bell triangle will therefore be

1

1 2

Following those same rules, we begin the third row with the last number in the previous row, or 2. Then we add that 2 to the number above it to find the next number is 3. Three plus the two above it makes five, so the row finishes with five.

1

1 2

2 3 5

Row four starts with the last of row three, i.e., 5.

Then $5+2=7$, so the next place is 7, and $7+3=10$, so the next digit is 10. Finishing this row and the next in the same manner, we get

1				
1	2			
2	3	5		
5	7	10	15	
15	20	27	37	52

The Bell number B_n is the number of partitions of a set of size n. We proved there that it satisfies the recurrence relation

$$B_n = \sum_{k=1}^{n} \binom{n-k}{k-1} B_{n-k}$$

with the convention that $B_0 = 1$. This recurrence is linear, but involves all the preceding terms, rather than a fixed number. There is no simple closed formula for B_n ,but there is a nice expression for its generating function, which we now derive. This is a type of generating function we haven't met before. The exponential generating function, or e.g.f., of the sequence $(a_0, a_1, ...)$ is the formal power series

$$\sum_{n \geq 0} \frac{a_n t^n}{n!}.$$

Proposition 3.3.1 *(Formula of Dobinski)*

$$B_n = \frac{1}{e} \sum_{k=1}^{+\infty} \frac{k^n}{k!}$$

proof. We proceed by strong recurrence .

That is true if $n = 0$ because

$$\frac{1}{e} \sum_{k=0}^{+\infty} \frac{k^0}{k!} = \frac{1}{e} \sum_{k=0}^{+\infty} \frac{1}{k!} = e$$

We give $m \geq 0$ and suppose that equalty is true for all n of $[0, m]$.

In these conditions

$$\begin{aligned} B_{m+1} &= \sum_{k=0}^{m} \binom{m}{n} B_n = \sum_{k=0}^{m} \binom{m}{n} (\frac{1}{e} \sum_{k=1}^{+\infty} \frac{k^n}{k!}) \\ &= \frac{1}{e} \sum_{k=0}^{+\infty} \frac{1}{k!} (\sum_{n=0}^{m} \binom{m}{n} k^n) = \frac{1}{e} \sum_{k=1}^{+\infty} \frac{1}{k!} (k+1)^m \end{aligned}$$

we can deduce:

$$\begin{aligned} B_{m+1} &= \frac{1}{e} \sum_{k=0}^{+\infty} \frac{(k+1)^{m+1}}{(k+1)!} = \frac{1}{e} \sum_{k=1}^{+\infty} \frac{k^{m+1}}{k!} \\ &= \frac{1}{e} \sum_{k=0}^{+\infty} \frac{k^{m+1}}{k!} \end{aligned}$$

(righ here $m + 1 \geq 1$ so we add 0 to the sum...) ■

Conclusion

This study of Stirling and Bell's numbers shown that Stirling regarded in first these numbers not as representing the number of partition and permutations but rather as the coefficients of a certain polynomial. We conclude that a simple experiments can give a great results.

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